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Exact Relativistic Charged Stellar Models with Anisotropic Stresss

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Abstract

In this work, we introduce a formal framework to derive interior solutions for the Einstein-Maxwell system, describing the configuration of a relativistic charged fluid sphere within the context of spherically symmetric gravitational fields, which are matched to the exterior Reissner-Nordstrom space–time. Building upon previous methodologies, particularly the innovative approach by Komathiraj and Sharma (2018), we refine the process by reducing the system to a recurrence relation with variable rational coefficients. This enables us to obtain a solution in terms of an infinite series. We demonstrate that it is possible to closed-form solutions within a specific range of model parameters. By impressive constraints on these variables, we reduce the outcomes to denote them in component of standard functions, strengthening their availability and understandability. The streamlined framework requires computing results through series analysis and solving resulting difference equations, at last providing formulations in the form of polynomials and products consisting of algebraic relations. Our study presents an extensive set of general solutions, providing understanding into the nature of charged fluid spheres in relativistic astrophysical context.

Keywords*:* Einstein-Maxwell system, Exact solutions, Differential equation, closed-form solution, Frobenius method, Recurrence formula

1. Introduction

General relativity is amongst the best theories of modern physics, which originated from simple intuition and emerges as a physical theory of prediction. It plays an important role in understanding the gravitational behavior of massive bodies in relativistic astrophysics. It has revolutionized the understanding of spacetime and the gravitational field. The interpretation of general relativistic phenomena results in the manifestation of the curvature of the spacetime produced by the presence of massive bodies. General relativity can be summed up in two statements. Firstly, spacetime is a curved pseudo-Riemannian manifold with an indefinite metric tensor field. Secondly, the relationship between the matter content and the curvature of spacetime is contained in the Einstein field equations. Shapiro Saul A Teukolsky et al in 1983 pointed out that general relativity models could be utilized in the study of strong gravitational fields and black holes for which the Newtonian gravity models are not suitable. We utilize the Riemann tensor for defining the curvature of spacetime and the energy momentum tensor for matter distributions and the contribution of the electromagnetic field in the presence of charge. The Einstein field equations link the curvature to the total energy momentum. The Bianchy identity ensures conservation of energy momentum.

In relativistic astrophysics, we construct mathematical models which describe the behavior of highly dense objects. These include models of gravitational collapse leading to black holes, neutron stars, pulsars and magnetars, characterized by extremely powerful magnetic fields,

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and extreme concentrations of energy such as gamma-ray bursts. Exact general relativistic models are relevant to study observed astrophysical processes. In this context spherically symmetric stellar models are important and they are extensively used in several applications. In astrophysics, the collapse of a stellar body can be accurately described by a spherically symmetric gravitational field as mentioned by Shapiro Saul A Teukolsky et al. Historically, the fundamental exact solutions found in relativistic astrophysics are listed as follows; The first exact solution to the Einstein field equations is the Schwarzchild exterior solution. The Schwarzchild interior solution describes the interior stellar gravitational field, with constant energy density. For small stars having low pressures, this approach gives an acceptable approximation. For a static spherically symmetric charged star, the Reissner-Nordstrom solution describes the exterior gravitational field. The Kerr solution is associated with the exterior of the body in rotating motion which is characterized by its mass and angular momentum. The Kerr-Newman solution is associated with the exterior of a charged rotating body. These exact solutions of the Einstein-Maxwell system are essential for describing astrophysical processes.

The Einstein-Maxwell system of equations is relevant in describing charged compact objects characterized by a strong gravitational field, such as neutron stars. The investigations of Ivanov, 2002 and Sharma et al, 2001demonstrate that the values of redshifts, luminosities and maximum mass of a compact relativistic star are affected by the presence of electromagnetic field and anisotropy. The works of Mak & Harko, 2004, Komathiraj & Maharaj, 2007a, Komathiraj & Maharaj, 2007b and Maharaj & Komathiraj, 2007 show the significant role of the electromagnetic field in describing the gravitational behavior of compact stars composed of quark matter. In the recent past, several scientists have tried to apply different approaches of finding solutions to the field equations. By a specific choice of the electric field, Hansraj & Maharaj, 2006 obtained solutions with isotropic pressures to the Einstein-Maxwell system. These solutions satisfy a barotropic equation of state and contain the Rej et al, 2021 model.

The Einstein–Maxwell equations are expressed in natural units where, and derivatives with respect to the radial coordinate are denoted by a prime (′). The energy density and pressure are defined relative to the comoving four-velocity of the fluid, while the electric field and charge density are incorporated using the electromagnetic energy–momentum tensor and the Maxwell equations. Overall, this study aims to derive and analyze new closed-form, physically meaningful solutions to the Einstein–Maxwell field equations. These models contribute to a deeper understanding of the role of charge and anisotropy in compact astrophysical bodies and help interpret the structure and stability of relativistic stellar configurations.

2. Methodology.

2.1 Einstein-Maxwell system

The line element for a static spherically symmetric spacetime, modelling the interior of the relativistic star, has the form,

$$ds^2 = -e^{-2v(r)}\,dt^2 + e^{-2\lambda(r)}\,dr^2 + r^2(d\theta^2 + sin^2\theta\,d\emptyset^2)\pi r^2 \tag{2.1}$$

where $v(r)$ and $\lambda(r)$ are arbitrary functions of the radial coordinate r. The Einstein–Maxwell field equations for the line element (2.1) are then obtained (in system of units having $8\pi G = c = 1$) as,

$$\frac{1}{r^2}\left(1 - e^{-2\lambda}\right) + \frac{2\lambda'}{r}\,e^{-2\lambda} = \rho + \frac{1}{2}E^2 \tag{2a}$$

$$\frac{-1}{r^2}\left(1 - e^{-2\lambda}\right) + \frac{2v'}{r}\,e^{-2\lambda} = p_r + \frac{1}{2} \tag{2b}$$

$$e^{-2\lambda}\left(v'' + v'^2 + \frac{v'}{r} - v'\lambda' - \frac{\lambda'}{r}\right) = p_t + \frac{1}{2}E^2 \tag{2c}$$

$$\frac{1}{r^2}e^{-\lambda}(r^2E)' = \rho + \frac{1}{2}E^2 \tag{2d}$$

The system incorporates the electric field intensity E and proper charge density σ through the energy-momentum tensor associated with the electromagnetic field and the Maxwell equations.

An alternative yet equivalent expression for the field equations can be formulated by introducing a new independent variable, denoted as x, and defining new functions y and Z as follows

$$x = Cr^2,\ A^2y^2(x) = e^{2v(r)},\ Z(x) = e^{2\lambda(r)} \tag{2.2}$$

Proposed by Durgapal & Bannerji, 1983where A and C are constants. Utilizing the transformation given in equation (2.2) the system (2) can be written in the following in the form as:

$$\frac{1-Z}{x} + \frac{2\lambda'}{r}\,e^{-2\lambda} = \frac{\rho}{C} + \frac{E^2}{2C} \tag{2.3a}$$

$$\frac{Z-1}{x} + 4Zx^2\ddot{y} = \frac{p_r}{C} - \frac{E^2}{2C} \tag{2.3b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1 - \frac{\Delta x}{C} - \frac{E^2x}{2C} = 0 \tag{2.3c}$$

$$\frac{4Z}{x}\left(x\dot{E} + E\right)^2 = \frac{\sigma^2}{C} \tag{2.3d}$$

Dots (·) in the equation (2.3) denote differentiation w.r.t the variable x. The system (2.3) establishes the gravitational characteristics of a charged perfect fluid with anisotropic pressure. Where $\Delta = p_t - p_r$ is the anisotropic factor or measure of anisotropy. As a result, we are presented with a linear system consisting of four independent equations involving seven unknown variables: ρ, p_r, p_t, E, σ, y and Z. The notable feature of this system is its capacity for obtaining a solution, achieved through appropriate substitutions of Z, E and Δ by integrating the second-order differential equation (2.3c), which happens to be linear in y.

2.2. Integration procedure

We study a particular form of the Einstein-Maxwell system (2.5.3) by making explicit choices for one of the gravitational potentials Z, electric field intensity E and the anisotropic factor Δ. For the metric function Z, we make the particular choice

$$Z = \frac{1+kx}{1+mx} \tag{2.2.1}$$

where k is a real constant. In (2.2.1) we take $k \neq 1$ to avoid negative energy densities which are unphysical for barotropic stars. The choice (2.2.1) was also made by Maharaj & Mkhwanazi, 1996 in their analysis of uncharged relativistic stars. Also observe that when $k = 0$ we regain the metric function considered by Hansraj & Maharaj, 2006 which generalizes the Finch and Skea neutron star model (1989). We have chosen the above form as it provides for a wider range of possibilities than the solutions of Hansraj & Maharaj, 2006, and it does produces charged and uncharged solutions which are necessary for a realistic model. Upon substituting (2.2.1) into (2.3c), the following is obtained

$$4(1+kx)(1+mx)\ddot{y} + 2(k-m)\dot{y} + \left[m(m-k) - \frac{(E^2+\Delta)(1+mx)^2}{Cx}\right] y = 0 \tag{2.2.2}$$

At this stage, we find it convenient to introduce the following transformation

$$z = 1 + mx \tag{2.2.3}$$

This transformation allows us to express the second-order differential equation (2.2.2) in a more straightforward and simplified manner.

$$4z(kz+m-k)\frac{d^2\tilde{y}}{dz^2} + 2(k-m)\frac{d\tilde{y}}{dz} + \left[(m-k) - \frac{\Delta}{C}\frac{Z^2}{Z-1}\frac{E^2z^2}{C(Z-1)}\right]\tilde{y} = 0 \tag{2.2.4}$$

Expressed in terms of the new dependent and independent variables, denoted as $\tilde{y} = y(z)$ and z respectively, integration of equation (2.2.4) requires the specification of the electric field intensity E and the anisotropic factor Δ. Although there exists a variety of choices for E and Δ, only a subset is deemed physically reasonable and capable of yielding closed-form solutions. To make (3.2.4) integrable, we choose the following particular forms for E and Δ,

$$\frac{E^2}{C} = \alpha\ \frac{(z-1)}{z^2} \quad \text{and} \quad \frac{\Delta}{C} = \beta\ \frac{(z-1)}{z^3} \tag{2.2.5}$$

Here, α and β are constants. The form of E^2 in equation (2.2.5) is considered physically acceptable as E remains regular and continuous throughout the sphere. It is important to note that $E = 0$ at $r = 0$. Various special cases of equation (2.2.5) have been previously examined by Takisa & Maharaj, 2013 and John & Maharaj, 2011, and they can also be reduced to the uncharged stellar model developed by Maharaj & Mkhwanazi, 1996. The form in (2.2.5) chosen for Δ ensures that anisotropy also vanishes at the center of the star (i.e. $p_r = p_t$ at the center). By substituting equation (2.2.5) into equation (2.2.4), we obtain

$$4z^2(kz+m-k)\frac{d^2\tilde{y}}{dz^2}+2z(k-m)\frac{d\tilde{y}}{dz}+\ [(m-k-\alpha)z-\ \beta\ ]\tilde{y}\ =0 \qquad (2.2.6)$$

This represents the master equation for the system of equations (2.5.3). When $\alpha = \beta = 0$, the differential equation (3.2.7) simplifies to:

$$4z\ \ (kz+m-k)\frac{d^2\tilde{y}}{dz^2}+2(k-m)\frac{d\tilde{y}}{dz}+\ [m-k]\tilde{y}\ =0 \qquad (2.2.7)$$

This represents the limiting condition and corresponds to an uncharged isotropic sphere.

2.3 General series solution

Obtaining a closed-form solution for the equation (2.2.6) proves challenging. However, it is possible to transform it into a differential equation suitable for integration using the Frobenius method. This transformation is carried out as follows, we introduce a new function u(z) such that

$$\tilde{y}\ =\ z^d u(z) \qquad (2.3.1)$$

Here, d represents a constant, and a similar type of transformation was previously employed by Komathiraj & Maharaj, 2007 in generating charged stellar models. With the aid of equation (2.3.1), the differential equation (2.2.6) can be reformulated as

$$4z(kz+m-k)\frac{d^2u}{dz^2}+2z[\ 4d(kz+m-k)+(k-m)\ ]\ \frac{du}{dz}-\{\ [k-m\ +\ \alpha-4kd(d-1)]z+2d(2d-3)(k-m)+\beta\ \}\ u\ =0 \qquad (2.3.2)$$

A significant simplification of the equation can be attained by setting

$$\beta\ =\ -[2d(2d-3)(k-m)] \qquad (2.3.3)$$

Consequently, Equation (2.3.2) transforms into

$$4z(z-c)\frac{d^2u}{dz^2}+2[\ 4d(z-c)+c\ ]\ \frac{du}{dz}-\ [c+\ \tilde{\alpha}-4d(d-1)]u\ =0 \qquad (2.3.4)$$

we have established

$$c=\frac{k-m}{k},\ k\tilde{\alpha}=\ \alpha \qquad (2.3.5)$$

Since the point $z = c$ is a special type of singular point known as a regular singular point for Equation (2.3.4), there are two solutions that can be expressed as power series centered around $z = c$. Consequently, we can represent the solution to the differential equation (3.3.4) using the Frobenius method.

$$u=\ \sum_{i=0}^{\infty}c_i(z-c)^{i+b}\ ,\ c_0\neq 0 \qquad (2.3.6)$$

where c_i represent the coefficients of the series and while b stands for a constant value.

To obtain a valid solution, it is essential to find both the coefficients c_i and the parameter b. By substituting expression (2.3.6) into the differential equation (2.3.4), we derive an equation known as the indicial equation,

$$2cc_0b(2b-1)=0$$

and the recurrence formula is

$$c_{i+1}=-\frac{\{4(i+b)(i+b-1+2d)-T\}}{2c(i+b+1)(2i+2b+1)}c_i \tag{2.3.7}$$

Considering $i \geq 0$ and $T = c+\tilde{\alpha}-4d(d-1)$, and given that $c_0 = 0$ and $c = k-m = 0$, we must choose $b=0\ or\ b=\frac{1}{2}$. The coefficients $c_1, c_2, c_3....$ can be expressed in relation to the initial coefficient c_0. This expression is given by

$$c_{i+1}=\prod_{p=0}^{i}-\frac{\{4(p+b)(p+b-1+2d)-T\}}{2c(p+b+1)(2p+2b+1)}c_0 \tag{2.3.8}$$

The expression (2.3.8) can be rigorously derived using the principle of mathematical induction. By applying this method, we can create two solutions that are linearly independent for Equation (2.3.4), utilizing expressions (2.3.6) and (2.3.8). When $b = 0$, the first solution is obtained as follows,

$$u_1=c_0\left[1+\sum_{i=0}^{\infty}\prod_{p=0}^{i}-\frac{\psi}{\eta}(z-c)^{i+1}\right]$$

Where,

$$\psi = 4p(p-1+2d)-[c+\tilde{\alpha}-4d(d-1)]\ and\ \eta = 2c(p+b+1)(2p+2b+1)$$

When $b=\frac{1}{2}$, the second solution is obtained as follows,

$$u_2=c_0(z-c)^{\frac{1}{2}}\left[1+\sum_{i=0}^{\infty}\prod_{p=0}^{i}-\frac{\alpha}{\beta}(z-c)^{i+1}\right]$$

Where $\alpha=(2p+1)(2p-1+2d)-[c+\tilde{\alpha}-4d(d-1)]\ and\ \beta=c(2p+3)(2p+2)$

Thus, we achieve a comprehensive solution for Equation (3.3.4) since the functions u_1 and u_2 exhibit linear independence. Expressing these functions in the original variable $x = Cr^2$, we obtain

$$y_1=c_0(1+mx)^d\left[1+\sum_{i=0}^{\infty}\prod_{p=0}^{i}-\frac{\psi_1}{\eta_1}\left[\frac{m(1+kx)}{k}\right]^{i+1}\right] \tag{2.3.9}$$

$$y_2=c_0(1+mx)^d\left[\frac{m(1+kx)}{k}\right]^{1/2}\left[1+\sum_{i=0}^{\infty}\prod_{p=0}^{i}-\frac{\alpha_1}{\beta_1}\left[\frac{m(1+kx)}{k}\right]^{i+1}\right] \tag{2.3.10}$$

The solutions y_1 and y_2 given in equations (2.3.9) and (2.3.10) respectively, generate the two linearly independent solutions to the differential equation (2.2.2) in terms of infinite series for appropriate values of the parameters k, m, α, and β. Therefore, the general solution to the differential equation (2.2.2), considering the specified electric field and anisotropic factor from equation (2.2.5), can be expressed as

$$y(x) = A_1 y_1(x) + A_2 y_2(x) \tag{2.3.12}$$

Here, A_1 and A_2 represent arbitrary constants and y_1 and y_2 are given in equations (2.3.9) and (2.3.10) respectively. By utilizing equation (2.5.3) and combining it with equation (2.3.12), we formulate the precise solution for the Einstein–Maxwell system in the following manner

$$e^{2\lambda} = \frac{(1+mx)}{1+kx} \tag{2.3.13a}$$
$$e^{2\nu} = A^2 y^2 \tag{2.3.13b}$$
$$\frac{\rho}{C} = \frac{(m-k)(3+m\dot{x})}{(1+mx)^2} - \alpha \frac{mx}{2(1+mx)^2} \tag{2.3.13c}$$
$$\frac{p}{C} = \frac{4(1+kx)}{(1+mx)}\frac{\dot{y}}{y} + \frac{k-m}{(1+mx)} + \alpha \frac{mx}{2(1+mx)^2} \tag{2.3.13d}$$
$$\Delta = p_t - p_r$$
$$p_t = \Delta + p_r \tag{2.3.13e}$$
$$\frac{E^2}{C} = \alpha \frac{mx}{(1+mx)^2} \tag{2.3.13f}$$
$$\frac{\Delta}{C} = \beta \frac{mx}{(1+mx)^3} \tag{2.3.13g}$$

Since the combined consideration of the metric function from equation (2.2.1) along with the electric field intensity and the anisotropic factor specified in equation (2.2.5) has not been explored before, as far as our knowledge extends, the set of solutions given by equation (2.3.13) has not been documented or reported before. An interesting aspect of this new family of solutions is that, by choosing $\alpha = 0$ and $\beta = 0$ ($d = 0$ *or* $d = 3/2$), we switch off the anisotropic factor in (2.3.13), then we regain the solutions for an isotropic charged case directly. Additionally, the solution provided in equation (2.3.13) is expressed in terms of a series involving real arguments, as opposed to complex arguments that may arise when utilizing mathematical software packages.

2.4 Terminating series

An engaging observation is that the series presented in equations (2.3.9) and (2.3.10) concludes for certain values of the parameters *k*, *m*, *α*, and *d*. This implies the potential to derive solutions in terms of elementary functions by imposing particular constraints on these parameters. Consequently, solutions can be expressed using polynomials and algebraic functions. In our approach, we opt to utilize the recurrence relation (2.3.7) instead of the series expressions (2.3.9) and (2.3.10) to obtain these elementary solutions.

2.4.1 Elementary solutions

By setting $b = 0$ in equation (2.3.7) and establishing that $c + \tilde{\alpha} - 4d(d-1) = 4n(n-1+2d)$ for integer values of n, we arrive at the following result,

$$c_{i+1} = 4\frac{(n-i)(n+i-1+2d)}{c(2i+1)(2i+2)} c_i \quad i \geq 0, \tag{2.4.1}$$

where n is a fixed integer, it's Obviously $c_{n+1} = 0$. As a result, the remaining coefficients c_{n+2}, c_{n+3}, c_{n+4} .... vanish. Solving Equation (2.4.1) leads to the following outcome,

$$c_i = \left(\frac{4}{c}\right)^i \frac{4n!(n+i-2+2d)!}{(2i)!\,(n-i)!\,(n-2+2d)!} c_0, \qquad 0 \leq i \leq n. \tag{2.4.2}$$

Using (2.3.6) (when b=0) and (2.4.2), we arrive at the following result,

$$u(z) = \sum_{i=0}^{n} \left(\frac{4}{c}\right)^{i} \frac{n!(n+i-2+2d)!}{(2i)!\,(n-i)!\,(n-2+2d)!}(z-c)^{i}, \tag{2.4.3}$$

where $c + \tilde{\alpha} - 4d(d-1) = 4n(n-1+2d)$.

By inserting b = $\frac{1}{2}$ into equation (2.3.7) and establishing that $c + \tilde{\alpha} - 4d(d-1) = (2n+1)(2n-1+4d)$, we arrive at the following result,

$$c_{i+1} = 4\frac{(n-i)(n+i+2d)}{c(2i+3)(2i+2)}c_i,\ i \geq 0 \tag{2.4.4}$$

where n is a fixed integer, it's Obviously $c_{n+1} = 0$. Consequently, the subsequent coefficients $c_{n+2}, c_{n+3}, c_{n+4}, \ldots$ vanish. Solving Equation (2.4.4) results in,

$$c_i = \left(\frac{4}{c}\right)^{i} \frac{n!(n+i-1+2d)!}{(2i+1)!\,(n-i)!\,(n-1+2d)!}c_0, \quad 0 \leq i \leq n. \tag{2.4.5}$$

Using (2.3.6) (when b = $\frac{1}{2}$) and (2.4.5), we arrive at the following result,

$$u(z) = c_0(z-c)^{\frac{1}{2}} \sum_{i=0}^{n} \left(\frac{4}{c}\right)^{i} \frac{n!(n_1+1)!}{(2i+1)!\,(n-i)!n_1!}(z-c)^{i} \tag{2.4.6}$$

where $n_1 = (n-1+2d)$ and $c + \tilde{\alpha} - 4d\,(d-1) = (2n+1)\,(2n-1+4d)$.

The expression represented by polynomial (2.4.3) and the product involving the polynomial and algebraic function (2.4.6) collectively produce a first solution for the given differential equation (2.3.4), provided that the parameters c, $\tilde{\alpha}$, and d assume appropriate values.

2.4.2 General solutions

Solutions to equation (2.3.4) can be derived by constraining the values of c, $\tilde{\alpha}$, and d in a manner that allows only elementary functions to remain. These elementary functions can be represented as polynomials and products of polynomials with algebraic functions. Employing expression (2.4.3), we articulate the initial set of general solutions to the differential equation (3.3.4) in the following format

$$u(z) = \sum_{i=0}^{n} \left(\frac{4}{c}\right)^{i} \frac{n!(n+i-2+2d)!}{(2i)!\,(n-i)!\,(n-2+2d)!}(z-c)^{i} \times \left[B_1 + B_2 \int \frac{z^{\left(\frac{1}{2}\right)-2d}(z-c)^{-1/2}}{\left[\sum_{i=0}^{n}\left(\frac{4}{c}\right)^{i}\frac{n!(n_2+1)!}{(2i)!\,(n-i)!n_2!}(z-c)^{i}\right]^{2}}dz\right] \tag{2.4.7}$$

Utilizing expression (2.4.6), the second category of general solutions for equation (2.3.4) is acquired in the following manner,

$$\mathrm{u(z)} = (z-c)^{\frac{1}{2}} \sum_{i=0}^{n} \left(\frac{4}{c}\right)^{i} \frac{n!\,(n_1+1)!}{(2i+1)!\;(n-i)!\,n_1!}(z-c)^{i} \times \\ \times \left[B_1 + B_2 \int \frac{z^{\left(\frac{1}{2}\right)-2d}(z-c)^{-\frac{3}{2}}}{\left[\sum_{i=0}^{n}\left(\frac{4}{c}\right)^{i} \frac{n!\,(n_1+1)!}{(2i+1)!\;(n-i)!\,n_1!}(z-c)^{i}\right]^{2}} dz\right] \tag{2.4.8}$$

Where $c + \alpha^{\sim} - 4d\,(d-1) = (2n+1)(2n-1+4d)$.

In equations (2.4.7) and (2.4.8), B_1 and B_2 represent integration constants. When expressed in terms of the original variable $x = Cr^2$, equation (2.4.7) can be reformulated in the following manner,

$$\mathrm{y(x)} = (1+mx)^{d} \sum_{i=0}^{n} \left(\frac{4k}{k-m}\right)^{i} \frac{n!(n_2+i)!}{(2i)!\,(n-i)!n_2!} \times \left[C_1 + C_2 \int \frac{(1+mx)^{\left(\frac{1}{2}\right)-2d}(1+kx)^{-\frac{1}{2}}}{\left[\sum_{i=0}^{n}\left(\frac{4k}{k-m}\right)^{i} \frac{n!(n_2+i)!}{(2i)!\,(n-i)!n_2!}t^{i}\right]^{2}} dx\right], \tag{2.4.9}$$

where $4 - m + \alpha - 4kd(d-1) = 4kn(n-1+2d)$

Equation (2.4.8) in terms of $x = Cr^2$, adopts the following representation,

$$\mathrm{y(x)} = C_3 \sum_{i=0}^{n} \left(\frac{4k}{k-m}\right)^{i} \frac{n!(n_1+1)!}{(2i+1)!\,(n-i)!n_1!} \times \left[C_1 + C_2 \int \frac{(1+mx)^{\left(\frac{1}{2}\right)-2d}(1+kx)^{-\frac{3}{2}}}{\left[\sum_{i=0}^{n}\left(\frac{4k}{k-m}\right)^{i} \frac{n!(n_1+i)!}{(2i+1)!\,(n-i)!n_1!}t^{i}\right]^{2}} dx\right], \tag{2.4.10}$$

Consequently, we have formulated two categories of solutions, denoted as (2.4.9) and (2.4.10), addressing the differential equation (2.2.2) under the assumed electric field and anisotropic factor (2.2.5). It's important to emphasize that these classes of solutions are versatile and can be applied to investigate stellar characteristics both in the presence and absence of charge and anisotropic pressure. By setting $\alpha = 0$ within expressions (2.4.9) and (2.4.10), solutions for an uncharged anisotropic sphere are obtained. Also, when $\beta = 0$ (resulting in $d = 0$ or $\frac{3}{2}$) (2.4.9) and (2.4.10), become charged isotropic models.

The solution in (2.4.9) and (2.4.10) have a simple form, and they have been expressed completely as combinations of polynomials and algebraic functions. This has the advantage of simplifying the investigation into the physical properties of charged dense stars with pressure anisotropy. We are now prepared to perform the integration of equations (2.4.9) and (2.4.10) with respect to specific values assigned to the parameters k, m, α, d, and n. When $n = 0$, equation (2.4.9) becomes,

$$\mathrm{y(x)} = (1+mx)^{d} \left[C_1 C_2 \int \frac{1}{(1+mx)^{2d-\left(\frac{1}{2}\right)}[(1+kx)]^{\frac{1}{2}}} dx\right], \tag{2.4.11}$$

where $k - m + \alpha - 4kd(d - 1) = 0$.

Similarly, when $n = 0$, equation (2.4.10) takes the following form,

$$y(x) = C_3 \left[C_1 + C_2 \int \frac{1}{(1+mx)^{2d-\left(\frac{1}{2}\right)}[(1+kx)]^{\frac{3}{2}}} dx\right], \quad (2.4.12)$$

where $2k - m + \alpha - 4kd^2 = 0$.

By choosing $k = -1/2$, $m = 1$, $\alpha = 0$ and $d = 3/2$ (or $\beta = 0$) within expression (3.4.11), we derive the following result,

$$y(x) = a_1(1+x)^{\frac{3}{2}} + a_2(2-x)^{\frac{1}{2}}(5+2x),$$

we make the assumptions $a_1 = C_1$ and $a_2 = -(2\ 2/27)C_2$. Consequently, we have successfully recovered the solution initially proposed by Durgapal & Bannerji, 1983. The were was Durgapal & Bannerji, 1983 model were Comprehensively studied and it was demonstrated that their Solutions produced a neutral be relativistic sphere that satisfied all physical criteria.

By choosing $k = 1/2$, $m = 1$, $\alpha = 0$ and $d = 0$ (or $\beta = 0$) within expression (2.4.12), we derive the following result,

$$y(x) = (2+x)^{\frac{1}{2}}\left(a_1 + a_2 \ln\left[(1+x)^{\frac{1}{2}} + (2+x)^{\frac{1}{2}}\right]\right) - a_2(1+x)^{1/2}$$

We make the assumptions $a_1 = C_1 / 2$ and $a_2 = 4C_2$. This particular set of solutions was previously identified by Maharaj & Mkhwanazi, 1996. The Maharaj & Mkhwanazi, 1996 model Satisfies all the physical conditions for an isolated spherically symmetric stellar source, and consequently has been utilized by many researchers to model neutral stars.

By choosing $k = 1/3$, $m = 1$, $\alpha = 1/3$ and $d = 0$ (or $\beta = 0$) within expression (2.4.12), we derive the following result,

$$y(x) = (3+x)^{\frac{1}{2}}\left(a_1 + a_2 \ln\left[(1+x)^{\frac{1}{2}} + (3+x)^{\frac{1}{2}}\right]\right) - a_2(1+x)^{1/2}$$

we make the assumptions $a_1 = C_1 / 3$ and $a_2 = 6C_2$ Notably, this specific solution aligns with the charged stellar model presented by John and Maharaj (2011). It's worth mentioning that a minor error identified in the John & Maharaj, 2011 has been rectified in this current work. The John & Maharaj, 2011 solution has been widely applied as a relativistic model for charged isotropic Stars with super dense matter.

3. Results and Discussion

Our aim in this thesis was to discover fresh exact solutions for the Einstein-Maxwell systems, specifically focusing on static spherically symmetric gravitational fields with a barotropic equation of state. We specifically opted for a linear equation of state linking energy density to radial pressure, with applications in modeling relativistic stars in astrophysical contexts. The solutions we've derived are believed to be physically plausible and could be instrumental in understanding dense stellar bodies within the framework of general relativity.

we make for physically sensible selections for the gravitational potential the intensity of the electric field and the anisotropic factor. These choices are made based on their respective form,

$$Z = \frac{1+kx}{1+mx}$$

$$\frac{E^2}{C} = \alpha \frac{mx}{(1+mx)^2}$$

$$\frac{\Delta}{C} = \beta \frac{mx}{(1+mx)^3}$$

We regain the findings of Komathiraj & Sharma, 2018and emphasize the presence of a singularity in the charge density at the center of the sphere. We observed that the condition of pressure isotropy can be written as,

$$4z^2(kz+m-k)\frac{d^2\tilde{y}}{dz^2} + 2z(k-m)\frac{d\tilde{y}}{dz} + [(m-k-\alpha)z - \beta\ ]\tilde{y} = 0$$

We've introduced a fresh method for generating a wide range of solutions to the Einstein-Maxwell system. This involves utilizing the transformation equations proposed by Durgapal & Bannerji, 1983 alongside a specific decay pattern for the electric field intensity. Many solutions obtained earlier are encompassed within our broader set of solutions. Additionally, we've illustrated that closed form solutions can be derived from the general series solution for specific model parameters. It's worth noting that the variety of solutions heavily relies on transformation: $\tilde{y} = z^d u(z)$. It's important to emphasize that while the integral forms represented by equations,

$$\mathrm{y(x)} = (1+mx)^d \sum_{i=0}^{n} \left(\frac{4k}{k-m}\right)^i \frac{n!\,(n_2+i)!}{(2i)!\ (n-i)!\ n_2!} \times \left[C_1 + C_2 \int \frac{(1+mx)^{\left(\frac{1}{2}\right)-2d}(1+kx)^{-\frac{1}{2}}}{\left[\sum_{i=0}^{n} \left(\frac{4k}{k-m}\right)^i \frac{n!\,(n_2+i)!}{(2i)!\ (n-i)!\ n_2!} \frac{m(1+kx)^i}{k}\right]^2} dx \right] \tag{4.0.2}$$

And

$$\mathrm{y(x)} = C_3 \sum_{i=0}^{n} \left(\frac{4k}{k-m}\right)^i \frac{n!(n_1+1)!}{(2i+1)!\,(n-i)!n_1!} \times \left[C_1 + C_2 \int \frac{(1+mx)^{\left(\frac{1}{2}\right)-2d}(1+kx)^{-\frac{3}{2}}}{\left[\sum_{i=0}^{n}\left(\frac{4k}{k-m}\right)^i \frac{n!(n_1+i)!}{(2i+1)!\,(n-i)!n_1!} t^i\right]^2} dx \right] \quad (4.0.3)$$

parameterized by *d*, don't immediately allow for the recovery of previously obtained charge-independent solutions, this can be achieved at a later point.

Once the integrations outlined in equations (4.0.2) and (4.0.3) are conducted and yield closed-form results for particular model parameters, both the charged and uncharged solutions expressed in elementary functions can be independently derived.

This method has been applied to recover the stellar solutions proposed by Durgapal & Bannerji, 1983 and Maharaj & Mkhwanazi, 1996, as well as the charged fluid solution by John & Maharaj, 2011. In conclusion, we've introduced two distinct closed-form solutions for the Einstein-Maxwell system. The simplicity of these solutions aids in effectively analyzing the physical behavior of a charged fluid sphere. Notably, our solution also yields a barotropic equation of state. However, it's important to note that closed-form solutions have only been generated for specific values of the model parameters. These solutions are suitable for modeling a charged relativistic sphere with anisotropic pressures. Notably, we've identified models that exhibit regularity at the center for the charge density, representing an advancement over the outcomes of Thirukkanesh & Maharaj, 2008.

4. Conclusion

In this project, we successfully derived new exact solutions to the Einstein-Maxwell system for static, spherically symmetric charged fluid spheres with anisotropic pressures. By employing a barotropic equation of state and using a linear relationship between energy density and radial pressure, we explored models relevant to relativistic stars. Our method, built upon the transformation proposed by Durgapal and Bannerji, allowed for the recovery of previously known solutions while also generating novel solutions. These closed-form solutions provide valuable insights into the physical behavior of charged compact objects, particularly in terms of regularity at the center, offering significant improvements over earlier models.

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